

# Digital Bandpass Beamforming With the Discrete Fourier Transform

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<p>A method is developed to form beam spectral estimates from a set of signals sampled from hydrophone outputs of an acoustic array. The signals must be frequency band limited over an interval <math>[W_o, W_o + W]</math> and sampled over a time interval <math>T</math>. A savings of <math>(W_o/W) + 1</math> in data storage and computation time is realized over conventional beam forming analysis methods employing sampling rates equal to twice the highest frequency in the signal band. Any set of delays <math>t_i</math> may be utilized and the beam-forming error determined as long as <math>t_i &lt; T/2</math>. The computed beam spectra can be made arbitrarily close to the actual beam spectra by choosing (a) a large number of points to be analyzed, (b) small input signal amplitudes at the end points of the time interval <math>T</math>, and (c) small actual spectrum amplitudes near <math>f=W_o</math> and <math>f=W_o+W</math>.</p> <p>This procedure would be particularly applicable to future sonar systems employing digital circuitry to beamform and spectrum analyze on-line.</p>			

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## ABSTRACT

A method is developed to form beam spectral estimates from a set of signals sampled from hydrophone outputs of an acoustic array. The signals must be frequency band limited over an interval  $[W_o, W_o + W]$  and sampled over a time interval  $T$ . A savings of  $(W_o/W) + 1$  in data storage and computation time is realized over conventional beamforming analysis methods employing sampling rates equal to twice the highest frequency in the signal band. Any set of delays  $t_i$  may be utilized and the beamforming error determined as long as  $t_i < T/2$ . The computed beam spectra can be made arbitrarily close to the actual beam spectra by choosing (a) a large number of points to be analyzed, (b) small input signal amplitudes at the end points of the time interval  $T$ , and (c) small actual spectrum amplitudes near  $f=W_o$  and  $f=W_o + W$ .

This procedure would be particularly applicable to future sonar systems employing digital circuitry to beamform and spectrum analyze on-line.

## PROBLEM STATUS

This is an interim report on the problem; work is continuing.

## AUTHORIZATION

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## DIGITAL BANDPASS BEAMFORMING WITH THE DISCRETE FOURIER TRANSFORM

### INTRODUCTION

The advent of the Fast Fourier Transform (FFT) has greatly facilitated computer calculations of power spectra. On-line spectrum analysis of signals by digital computer is expected to play an important role in future sonar systems. The method presented here allows a beamformed spectrum to be computed from a set of appropriately sampled bandpass signals representing hydrophone outputs of an array. If the ratio of the center frequency or carrier to the signal band about the carrier is much greater than one, a conventional beamforming analysis utilizing sampling of the data at twice the highest frequency in the signal band not only wastes time and money in providing many spectral estimates at frequencies of no interest, but requires inordinately large amounts of storage to obtain a desired resolution. These high sampling rates are normally used when implementing a digital beamforming analysis in order to obtain small, discrete time intervals for delaying purposes. A bandpass sampling method of beamforming would be more efficient, would yield better resolution for the same amount of input data samples, and would result in good coverage of the useful bandwidth utilizing all the available spectral estimates. Such a scheme is employed here with the following results:

- (a) The beamformed spectrum is expressed as a single equation utilizing the conventional Discrete Fourier Transform (DFT).
- (b) The sampling rates are considerably lower than those normally employed for digital beamforming. Therefore, a substantial saving of data storage and computation time results.
- (c) Sampling the signal at these low rates does not constrain the delays  $t_i$  to be integral multiples of the sampling interval and, in fact, any delay is realizable with this method and an associated beamforming error may be calculated as long as  $t_i < T/2$  where  $T$  is the signal length to be analyzed.
- (d) The use of first-order sampling\* affords a very simple and practical means of obtaining the discrete data samples for analysis.

### DIGITAL BANDPASS SPECTRUM ANALYSIS

In this section we review the concept of bandpass sampling and develop a method for estimating the power spectrum of a bandpass function  $g(t)$ . The results of this section are then used in the next section to develop and construct the beam-spectrum equation.

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\*A sampling is defined as first-order if the sample points are equispaced. See, for example, Ref. 1.

Consider a function  $g(t)$  which is band limited to the frequency interval  $[W_o, W_o+W]$ . The function  $g(t)$  may be reconstructed exactly (2) from an infinite set of first-order samples taken at a rate  $2W$  only if  $W_o = CW$ , where  $C = 1, 2, 3, \dots$ . Although higher order sampling permits reconstruction of the original signal for less severe constraints on  $W_o$  and  $W$ , we will only consider the case for which

$$W_o = CW, C = 1, 2, 3, \dots \quad (1)$$

since first-order sampling is more amenable to current operational digitizing systems. The complexity of higher order sampling schemes makes first-order sampling a very simple and practical means of obtaining the discrete data samples. In a practical situation for which  $W_o$  is not an integer multiple of the signal bandwidth,  $W$  can be easily chosen slightly larger than the actual signal bandwidth so that Eq. (1) is satisfied.

Let samples of  $g(t)$  be taken at a rate  $2W$  and let  $W_o = CW$ , where  $C$  is a positive integer. The conventional Nyquist rate for sampling a function which is band limited to the interval  $[0, W_o+W]$  is equal to  $2(W_o+W)$ . Thus, by sampling at a rate  $2W$  ("submultiple" or "band-pass" sampling), a factor of  $(W_o/W)+1 = C+1$  savings in data storage is realized. We define the reconstruction  $g_R(t)$  of  $g(t)$  as in (2) by

$$g_R(t) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \left\{ 2(W_o+W) \operatorname{sinc} \left[ 2(W_o+W) \left( t - \frac{n}{2W} \right) \right] - \right. \\ \left. 2W_o \operatorname{sinc} \left[ 2W_o \left( t - \frac{n}{2W} \right) \right] \right\} \quad (2)$$

where

$$\operatorname{sinc} X = \frac{\sin \pi X}{\pi X}.$$

It can be shown that  $g_R(t)$  is equal to  $g(t)$ , exactly, for all  $t$ .

In addition to being band limited, let  $g(t)$  be time limited to the interval  $[0, T]$ . There are now  $2WT = N$  samples obtained from sampling  $g(t)$  in the interval  $[0, T]$ . The reconstruction, Eq. 2, for  $g(t)$  remains the same, except that the limits on the sum are changed to  $n = 0$ , and  $n = (N-1)$ . Even though a function cannot strictly be both band limited and time limited, these conditions may be met approximately (3). From this point we will consider only those  $g(t)$  whose values are negligible outside some time interval of length  $T$  and whose spectrum values are also negligible outside the frequency interval  $[W_o, W_o+W]$ . The function  $g_R(t)$  is now approximately equal to  $g(t)$  over the interval  $T$ . The closeness of the approximation depends on the smallness of the spectrum of  $g(t)$  outside the interval  $[W_o, W_o+W]$  and the smallness of  $g(t)$  outside  $T$  (see App. A).

Define the Fourier transform pair to be

$$\begin{aligned}\psi[g(t)] &\equiv G(f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi ift} dt \\ \psi^{-1}[G(f)] &\equiv g(t) = \int_{-\infty}^{\infty} G(f)e^{2\pi ift} df\end{aligned}\tag{3}$$

where  $\psi$  denotes the transform operator and  $f$  is measured in Hertz. Two transforms which will be used in the development are

$$\psi[\text{sinc}(Xt)] = \begin{cases} \frac{1}{X}, & |f| \leq \frac{X}{2} \\ 0, & |f| > \frac{X}{2} \end{cases}$$

and

$$\psi[g(t+\alpha)] = e^{2\pi if\alpha} \psi[g(t)] = e^{2\pi if\alpha} G(f).$$

We now proceed to derive an expression for the transform of the bandpass sampled time function  $g(t)$  in terms of the Discrete Fourier Transform (DFT). First, we apply the operator  $\psi$  to  $g_R(t)$  to obtain

$$G_R(f) \equiv \psi[g_R(t)] = \begin{cases} \frac{1}{2W} \sum_{n=0}^{N-1} \left( g\left(\frac{n}{2W}\right) e^{-2\pi ifn/2W} \right), & W_o \leq |f| \leq W_o + W \\ 0, & \text{elsewhere} \end{cases} \tag{4}$$

Equation (4) expresses the transform of  $g_R(t)$  as a function of continuous frequency  $f$ . However, the computer operates on a discrete set of data, i.e., on the values  $g(n/2W)$  for  $n = 0, 1, \dots, N-1$ . The DFT resulting from the operation consists of  $N$  output values at the frequencies

$$f_k = \frac{k}{T} = \frac{2W}{N} k$$

where  $k$  is an integer satisfying the relation  $W_o T \leq |k| \leq (W_o + W)T$ .



Substitution of  $f_k$  for  $f$  in Eq. (4) yields

$$G_R(f_k) = \begin{cases} \frac{T}{N} \sum_{n=0}^{N-1} \left( g\left(\frac{nT}{N}\right) e^{-2\pi i k n/N} \right), & W_o \leq |f_k| \leq W_o + W \\ 0, & \text{elsewhere} \end{cases} \quad (5)$$

where  $f_k = k/T$ .

The sufficiency conditions that enable the function  $G_R(f_k)$  to be made arbitrarily close to the "correct" transform  $G(f)$ , evaluated at  $f = f_k$ , are shown in App. A. These conditions show the amount of data, the time domain weighting, and the degree of bandpass filtering that is necessary to meet a desired degree of accuracy in  $G_R(f_k)$ .

The form of Eq. (5) looks much like the DFT. However, the DFT is usually computed for  $k = 0, 1, 2, \dots, N-1$ . The bounds on the index  $k$  in Eq. (5) are

$$W_o T \leq |k| \leq (W_o + W)T.$$

Since  $2WT = N$  and  $W_o = CW$ , we have

$$\frac{CN}{2} \leq |k| \leq \frac{(C+1)N}{2}. \quad (6)$$

Therefore, adjustments must be made in order to properly evaluate the spectrum of  $g(t)$  by means of the conventional DFT.

Since  $C$  is an integer, we can write

$$C = 2m - \ell$$

where  $m$  is a positive integer and  $\ell = 0$  or  $1$  as  $C$  is even or odd.

If we introduce a new index  $k'$  such that

$$k = \begin{cases} (CN/2) + k' = mN - (\ell N/2) + k', & k' = 0, 1, \dots, (N/2)-1 \\ -(CN/2) + k' = -mN + (\ell N/2) + k', & k' = -N/2, \dots, -2, -1 \end{cases}, \quad (7)$$

then

$$\frac{CN}{2} \leq |k| \leq \frac{(C+1)N}{2} \text{ for } 0 \leq |k'| \leq N/2$$

Substituting Eq. (7) into Eq. (5) and noting that  $e^{-2\pi i k' n/N}$  is periodic in  $k'$  with period  $N$ , and that  $e^{-\pi i n \ell} = e^{+\pi i n \ell}$ , we have

$$G_R(f_k) = \begin{cases} \frac{T}{N} \sum_{n=0}^{N-1} \left( g\left(\frac{nT}{N}\right) e^{\pi i n \ell} e^{-2\pi i n k'/N} \right), & k' = 0, 1, \dots, N-1 \\ 0, & \text{all other } k' \end{cases} \quad (8)$$

where

$$f_k = \begin{cases} \frac{CN}{2T} + \frac{k'}{T}, & k' = 0, 1, 2, \dots, (N/2)-1 \\ -\frac{(C+2)N}{2T} + \frac{k'}{T}, & k' = N/2, \dots, N-1 \end{cases}.$$

In order to reduce Eq. (8) to the form of the DFT we must consider the two cases for  $C$  even or odd. When  $\ell = 0$  ( $C$  is even) Eq. (8) reduces to

$$G_R(f_k) = \begin{cases} \frac{T}{N} \sum_{n=0}^{N-1} \left( g\left(\frac{nT}{N}\right) e^{-2\pi i n k'/N} \right), & k' = 0, 1, \dots, N-1 \\ 0, & \text{all other } k' \end{cases} \quad (9)$$

where

$$f_k = \begin{cases} \frac{mN+k'}{T}, & k' = 0, 1, \dots, (N/2)-1 \\ -\frac{mN-N+k'}{T}, & k' = N/2, \dots, N-1 \end{cases}.$$

This is simply the equation for the DFT of  $g(nT/N)$ , so that

$$G(f_k) \approx G_R(f_k) = \text{DFT}_{k'} \left[ g\left(\frac{nT}{N}\right) \right], \quad k' = 0, 1, \dots, N-1. \quad (10)$$

Thus for  $C$  an even integer, the "correct" Fourier transform of the bandpass function  $g(t)$  sampled over the interval  $T$  may be approximated by means of the DFT of the  $N$  bandpass samples of  $g(t)$ . Figure 1 shows graphically a spectrum in the three stages of processing: (a) the original spectrum  $G(f)$  for  $C$  an even integer; (b) the spectrum of  $G(f)$  after sampling at a rate  $2W$ ; and (c) the spectrum obtained from the DFT of the  $N$  sampled values of  $g(t)$ .

When  $\ell = 1$  ( $C$  is odd) we find\*

$$G_R(f_k) = \begin{cases} \frac{T}{N} \sum_{n=0}^{N-1} \left( g\left(\frac{nT}{N}\right) e^{\pi i n} e^{-2\pi i n k'/N} \right), & k' = 0, 1, \dots, N-1 \\ 0, & \text{all other } k' \end{cases}$$

$$= \text{DFT}_{k'} [(-1)^n g(nT/N)]. \quad (11)$$

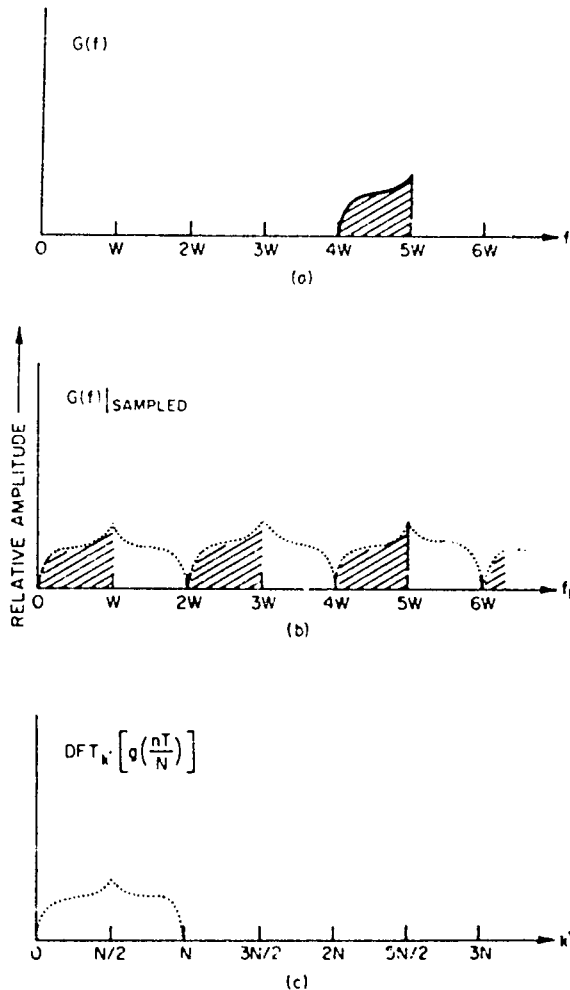


Fig. 1—The ideal spectrum  $G(f)$  for an even value of  $C$  is illustrated in (a). After sampling at a rate  $2W$ , the spectrum becomes as shown in (b). Finally, the spectrum obtained from the Discrete Fourier Transform (DFT) of the  $N$  sampled values of  $g(t)$  is shown in (c). These three sketches show the amplitude spectrum in three stages of processing.

Thus for  $C$  odd, the data may be spectrum analyzed using the DFT if the sign of every other data point is first changed. If the percentage bandwidth of the signal is small enough, it may be advantageous to increase  $W$  slightly to make  $C$  even. Then the  $(-1)^n$  term can be avoided.

\*The factor  $e^{\pi i n} = (-1)^n$  is equal to  $e^{2\pi i n t W}$  where  $t = nT/N$  and  $W = N/2T$ , corresponding in the frequency domain to a shift of  $W$ ; i.e.,

$$\psi[e^{2\pi i n t W} g(t)] = G(f-W).$$

The Inverse Discrete Fourier Transform (IDFT) of  $G_R(f_k)$  for  $C$  even yields the original time series  $g(nT/N)$ . The function  $g_R(t)$  may then be formed by using the appropriate sampling interpolation formula. The IDFT of  $G_R(f_k)$  for  $C$  odd yields the time series  $(-1)^n g(nT/N)$ . The function  $g_R(t)$  may be formed by first changing the sign of every other data value and then applying the sampling interpolation formula.

The power density spectrum  $S(f_k)$  is formed from the Fourier transform of  $g(t)$  by letting

$$S(f_k) = \frac{1}{T} G_R(f_k) G_R^\dagger(f_k)$$

where  $\dagger$  denotes the complex conjugate.

Thus, in this section we found that the Fourier transform of a function  $g(t)$ , appropriately time- and frequency-band limited, can be computed from a finite set of bandpass sampled data of  $g(t)$  by means of the DFT, and at a computer storage savings proportional to the ratio of the highest frequency in the signal band to the bandwidth of the signal. If the number of data samples  $N$  is a power of two, the FFT may be employed to calculate the DFT.

It now remains to incorporate an arbitrary delay into  $g(t)$  for the purpose of beamforming. This will be done in the next section.

## DIGITAL BANDPASS BEAMFORMING

The previous section presented and developed some properties of bandpass functions relating to the DFT. Once these methods are at hand, it is a fairly simple matter to incorporate a time delay  $t_o$  into the equations and develop an expression for the transform of the delayed function  $g(t-t_o)$ . After doing this, the expression will be generalized to include a set of signals, each with its own delay, representing the outputs of hydrophones in an array to be beamformed and spectrum analyzed.

Basically, a beam is formed from signals which are delayed by predetermined values and then summed. These delays can be introduced in either the time domain or the transform (frequency) domain of the signal because of the relation

$$g(t-t_o) = \psi^{-1} [e^{-2\pi i f t_o} G(f)]$$

where

$$G(f) = \psi[g(t)]. \quad (12)$$

Introduction of a time delay  $t_o$  in Eq. (4) results in

$$G_R^d(f) = \psi[g_R(t-t_o)] = \begin{cases} \frac{1}{2W} e^{-2\pi i f t_o} \sum_{n=0}^{N-1} \left( g\left(\frac{nT}{N}\right) e^{-2\pi i f n/2W} \right), & W_o \leq |f| \leq W_o + W \\ 0, & \text{elsewhere} \end{cases} \quad (13)$$

where  $G_R^d(f)$  is the transform of the delayed reconstruction function.

Now let  $f = f_k = k/T$ , as in the previous section. Then,

$$G_R^d(f_k) = \begin{cases} \frac{T}{N} e^{-2\pi i t_o k/T} \sum_{n=0}^{N-1} \left( g\left(\frac{nT}{N}\right) e^{-2\pi i k n/N} \right), & \frac{CN}{2} \leq |k| \leq \frac{(C+1)N}{2} \\ 0, & \text{all other } k \end{cases} \quad (14)$$

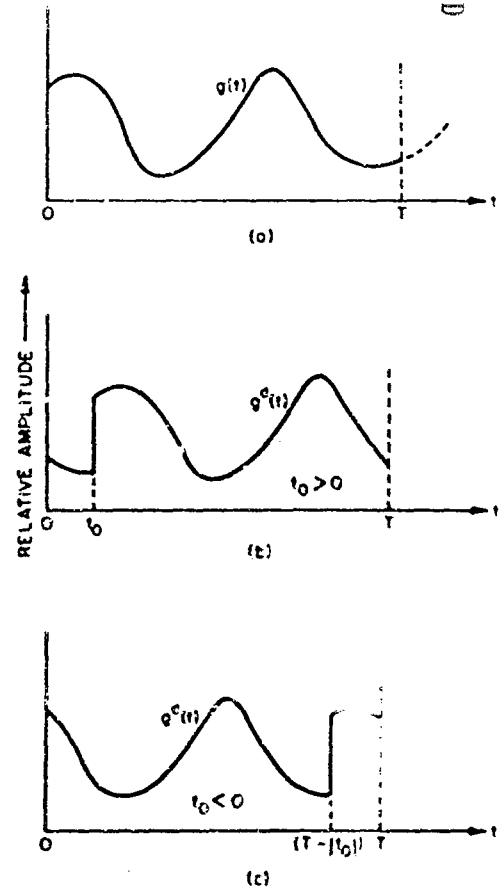
It is clear from Eq. (14) that we are forming the transform  $G_R^d(f_k)$  of the delayed reconstruction function from sampled values of  $g(t)$  for  $0 \leq t \leq T$ . However, the nature of the sampling process is such that  $g(nT/N)$  appears to be cyclic in  $n$  with period  $N$ , and  $G_R(f_k)$  is cyclic in  $k$  (see previous section) with period  $N$ . A delay of  $t_o$  will therefore cause the function  $g(t)$ ,  $0 \leq t \leq T$ , to be delayed into the new function (see Fig. 2)

$$\text{or } g^d(t) = \begin{cases} \left. \begin{aligned} &g(t-t_o), & t_o \leq t \leq T \\ &g(T-t_o+t), & 0 \leq t \leq t_o \end{aligned} \right\} & \text{for } t_o > 0 \\ \left. \begin{aligned} &g(t+|t_o|), & 0 \leq t \leq (T-|t_o|) \\ &g(t-T+|t_o|), & (T-|t_o|) \leq t \leq T \end{aligned} \right\} & \text{for } t_o < 0 \end{cases} \quad (15)$$

Therefore, those values of the delayed reconstruction function  $g_R(t_n-t_o)$  computed for the interval  $[0, t_o]$  for  $t_o > 0$ , or for the interval  $[T-|t_o|, T]$  for  $t_o < 0$ , may not be accurate since this part of the function has been "folded over" or "cycled" from the other end of  $g(t)$ . However, if the quantity  $t_o$  is much less than  $T$ , any error in the delayed function  $g_R(t_n-t_o)$  on the interval  $[0, |t_o|]$  or  $[T-|t_o|, T]$  will detract negligibly from the correctness of the function on the interval  $[0, T]$  (see App. B). In fact, for most practical acoustic underwater array problems,  $t_o$  will be on the order of milliseconds and  $T$  will be several seconds, so that  $(t_o/T) \approx 10^{-2}$ . An error equation is developed in App. B which includes the effects from "cyclic" delaying. It is shown in App. C that in at least one specific case the cyclic delaying error is small for delays close to  $T/2$ .

To reduce Eq. (14) to the form of the DFT, we proceed as in the previous section. Since  $C$  is an integer we can write

Fig. 2—A delay of  $t_0$  will cause the time-limited function  $g(t)$  [sketch (a)] to be delayed into the new function  $g^d(t)$ , for  $t_0 > 0$  [sketch (b)] or for  $t_0 < 0$  [sketch (c)].



$$C = 2m - \ell$$

where  $m$  is a positive integer and  $\ell = 0$  or  $1$  as  $C$  is even or odd. Introduce a new index  $k'$  such that

$$k = \begin{cases} mN - \frac{\ell N}{2} + k', & k' = 0, 1, 2, \dots, (N/2) - 1 \\ -mN + \frac{\ell N}{2} + k', & k' = -N/2, \dots, -2, -1 \end{cases}$$

We substitute  $k'$  for  $k$  in Eq. (14) and then specialize to the two cases of even or odd  $C$ .

For  $\ell = 0$  ( $C$  even) we find

$$G_R^d(f_k) = \left\{ \begin{array}{ll} \frac{T}{N} e^{\frac{-2\pi i t_0}{T} (mN + k')} \text{DFT}_{k'} \left[ g\left(\frac{nT}{N}\right) \right], & k' = 0, 1, \dots, (N/2)-1 \\ \frac{T}{N} e^{\frac{-2\pi i t_0}{T} (-(m+1)N + k')} \text{DFT}_{k'} \left[ g\left(\frac{nT}{N}\right) \right], & k' = N/2, \dots, N-1 \\ 0, & \text{all other } k' \end{array} \right\}. \quad (16)$$

Thus for  $C$  even, the transform of the delayed function  $g(t-t_0)$  is approximated by computing the DFT of the set of values  $g(nT/N)$  and then multiplying the  $k'$ th value by the appropriate exponential phase factor.

For  $\ell = 1$  ( $C$  odd) we have

$$G_R^d(f_k) = \left\{ \begin{array}{ll} \frac{T}{N} e^{\frac{-2\pi i t_0}{T} (mN - (N/2) + k')} \text{DFT}_{k'} \left[ (-1)^n g\left(\frac{nT}{N}\right) \right], & k' = 0, \dots, (N/2)-1 \\ \frac{T}{N} e^{\frac{-2\pi i t_0}{T} (-mN - (N/2) + k')} \text{DFT}_{k'} \left[ (-1)^n g\left(\frac{nT}{N}\right) \right], & k' = N/2, \dots, N-1 \\ 0, & \text{all other } k' \end{array} \right\}. \quad (17)$$

Thus for  $C$  odd, the transform of the delayed function  $g(t-t_0)$  is approximated by computing the DFT of the set of values  $(-1)^n g(nT/N)$  and multiplying the  $k'$ th value by the appropriate exponential phase factor.

Equation (16) or (17) allows us to approximate the Fourier transform of a delayed time function  $g(t-t_0)$  by operating on bandpass samples of  $g(t)$  taken from  $t = 0$  to  $t = T$ . The only new restriction is that the delay  $t_0$  should be small compared to the length of signal  $T$  to be analyzed.

In order to generalize the equations for the beamforming case, consider an array of  $p$  hydrophones, in which the  $r$ th hydrophone receives a signal  $g^r(nT/N)$ ,  $r = 1, 2, \dots, p$ . Let the delay to be introduced into the  $r$ th received signal be  $t_r$  and let  $(t_r)_{\max}/T$  be much less than unity. For  $C$  an even integer the transform of the formed beam (computed from all receiver output signals 1 to  $p$ , each having been sampled at a rate  $2W$  from 0 to  $T$ ) is approximated by

$$G_R^{\text{Beam}}(f_k) = \left\{ \begin{array}{ll} \frac{T}{N} \sum_{r=1}^p \left\{ e^{\frac{-2\pi i t_r}{T} (mN+k')} \text{DFT}_{k'} \left[ g^r \left( \frac{nT}{N} \right) \right] \right\}, & k' = 0, 1, \dots, (N/2)-1 \\ \frac{T}{N} \sum_{r=1}^p \left\{ e^{\frac{-2\pi i t_r}{T} (-(m+1)N+k')} \text{DFT}_{k'} \left[ g^r \left( \frac{nT}{N} \right) \right] \right\}, & k' = N/2, \dots, N-1 \\ 0, & \text{all other } k' \end{array} \right\}. \quad (18)$$

This is the final result of the analysis. The beam spectrum is readily calculated by means of Eq. 18 which utilizes the DFT and the bandpass sampled data from the individual hydrophones in the array. Equation (18) could be implemented with a digital computer employing the Fast Fourier Transform to compute the DFT of the input signals. To save even more computation time, only those Fourier estimates at frequencies  $f_{k'}$  of interest need be multiplied by the exponential phase factors and summed. A wide choice of analysis "windows" is available (3), and incorporation of such a function in Eq. (18) would serve the dual purpose of reducing spectral leakage and providing the necessary time-limiting approximations on  $g(t)$ .

## SUMMARY

Signals which are band limited to the frequency interval  $[W_o, W_o + W]$  can be sampled from  $t = 0$  to  $t = T$  at a rate  $2W$  and digitally processed according to Eq. (18) to obtain a beamformed spectrum of the inputs. A savings of  $(W_o/W)+1$  in data storage or an improved frequency resolution is realized over the conventional digital beamforming methods because of bandpass sampling. The computed spectrum can be made arbitrarily close to the actual spectrum by choosing a large number of points analyzed, small input signal amplitudes near  $t = 0$  and  $t = T$ , and small spectrum values near  $f = W_o$  and  $f = W_o + W$ . The delays  $t_i$  are not constrained to be integral multiples of the sampling interval  $\Delta t$ . Any set of delays  $t_i$  may be used and a beamforming error term calculated as long as  $t_i < T/2$ . This method could be used for digital on-line analysis of narrow-band signals constituting outputs of a receiving array in a sonar system, or it might be used as a time- and space-saving data processing technique employing a conventional digital computer.

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## APPENDIX A

## BANDPASS SPECTRUM ERROR

There are five sources of error which contribute to the total error in the beamformed amplitude spectrum  $G_R^{\text{Beam}}(f_k)$ . Only one of these errors originates from the delay properties of the processing. The rest are present in the Discrete Fourier Transform (DFT) regardless of any delay introduced. The five errors are

- a. spectrum aliasing error, present because of  $G(f)$  is not strictly bandlimited to  $W$ ,
- b. truncation error from the time domain caused by sampling  $g(t)$  only over the interval  $T$ ,
- c. spectral leakage through sidelobes of the analyzing window,\*
- d. error from spectrum curvature, and
- e. a time domain cycling error caused by the cyclic delaying properties of the beamforming process.

In this appendix we discuss the first four errors, which contribute to the total error in the bandpass spectrum computed via the DFT. (In App. B we will discuss the error in the *beamformed* bandpass spectrum.) The errors from leakage and curvature are discussed qualitatively; the aliasing and truncation errors will then be accounted for quantitatively by combining their effects into a single error formula.

The degree of spectral leakage that can be tolerated is largely dependent on the characteristic of the spectra to be analyzed. If the spectra are relatively flat in the band of interest, leakage of energy through sidelobes of the analyzing spectral window would not be of concern. But for spectra with a large dynamic range of amplitudes, or for one with sharp separated spikes, leakage can result in false estimates of power. If leakage is a concern, the Hanning window is an excellent smoothing window which may be applied either as a multiplication of the time data or as a convolution of the complex Fourier amplitude spectrum (3). Although the resolution is reduced, the peak level of the Hanning spectral window at the first sidelobe is more than 30 dB down from the peak level at the main lobe.

A restriction on the curvature of the spectrum is made by requiring that the energy in the ideal spectrum be almost constant over the analyzing window width. Here "analyzing

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\*The analyzing window may be thought of as a spectral weighting function of unity gain which is convolved with the ideal spectrum  $G(f)$  to obtain the estimates of spectral amplitude.

“window width” refers to the width of the main lobe of the spectral analyzing window. This requirement assures that an unbiased estimate of the true spectrum will be obtained (Ref. 3, p. 8). A user must choose an analysis time  $T$  long enough to assure enough resolution or minimal window width.

Now, to obtain the quantitative estimate of the error from aliasing and truncation, we follow the procedure developed by Moseley. Moseley calculated these errors in the DFT of a time function  $g(t)$  sampled over the interval  $T$ . The spectrum of  $g(t)$  in his development is low pass and is bandlimited to a frequency  $W$  (Ref. 3, App. B). We will use the results of his analysis to obtain a measure of the aliasing error and then to obtain a measure of the truncation error for our bandpass spectrum. In order to apply his results to the bandpass case we proceed to split the spectrum into two parts. This enables us to compute the error in the DFT caused by aliasing. Let

$$G(f) = G'(f) + G''(f) \quad (A1)$$

where

$$G'(f) = \begin{cases} G(f), & W_0 \leq f \leq W_0 + W \\ \approx 0, & \text{elsewhere} \end{cases}$$

and

$$G''(f) = \begin{cases} G(f), & -W_0 - W \leq f \leq -W_0 \\ \approx 0, & \text{elsewhere} \end{cases}.$$

Recall that

$$G(f) = \begin{cases} G(f), & W_0 \leq |f| \leq (W_0 + W) \\ \approx 0, & \text{elsewhere} \end{cases}.$$

We proceed to find the aliasing error in  $G'(f)$  and  $G''(f)$  and then add these together to obtain the error in  $G(f)$ .

Consider a frequency translation of  $G'(f)$  to  $G'(f')$ , where  $f' = f - W_0 - (W/2)^*$ . Then

$$G'(f') = \begin{cases} G'(f'), & 0 \leq |f'| \leq W/2 \\ \approx 0, & W/2 \leq |f'| \leq \infty \end{cases}. \quad (A2)$$

\*This frequency shift merely amounts to a complex phase multiplication of the time data.

Assume that  $|G'(f')|$  approaches zero at least as rapidly as

$$\frac{\epsilon}{|f'|^\alpha} \quad \text{for} \quad W/2 \leq |f'| < \infty \quad (\text{A3})$$

where  $\alpha > 1.0$  and  $\epsilon$  is some small constant. Denoting the aliasing error by  $\gamma_1$ , Moseley calculates this error in  $G'(f')$  to be given by

$$|\gamma_1| < \frac{|\epsilon| 2^{\alpha+1} \zeta(\alpha)}{(SF)^\alpha} \quad (\text{A4})$$

where  $SF$  is the sampling frequency ( $2W$  for our case), and  $\zeta(\alpha)$  is the Riemann zeta function defined by

$$\zeta(\alpha) = \sum_{\ell=1}^{\infty} \ell^{-\alpha}.$$

Similarly, after translating  $G''(f)$  to  $G''(f'')$ , where  $f'' = f + W_0 + (W/2)$ , we find that the aliasing error in  $G''(f'')$  is also less than  $\gamma_1$ . This time we require that  $|G''(f'')|$  approach zero at least as rapidly as

$$\frac{\epsilon}{|f''|^\alpha} \quad \text{for} \quad W/2 \leq |f''| < \infty.$$

For our bandpass spectrum, the restrictions on  $G'(f')$  and  $G''(f'')$  require that  $G(f)$  approach zero at least as rapidly as

$$\frac{\epsilon}{|f| - W_0 - (W/2)^\alpha} \quad \text{for} \quad |f| \text{ outside } [W_0, W_0 + W].$$

Then these errors may be added to obtain an approximate estimate of the aliasing error in the bandpass spectrum:

$$|\gamma_1^{\text{bp}}| < \frac{|\epsilon| 2^{\alpha+2} \zeta(\alpha)}{(SF)^\alpha}. \quad (\text{A5})$$

This error is twice the error in the low-pass spectrum bandlimited to  $W$  because for the bandpass case we have two frequency cutoff regions, while for the low-pass case there is only one. While there are only  $C$  aliased bands from frequencies less than  $W_0$ , and an infinite number of aliased bands from frequencies above  $W_0 + W$ , the falloff of the out-of-band energy is usually fast enough so that doubling the low-pass aliasing error is a good approximation to the aliasing bandpass error. In any case the estimated bandpass aliasing error is a valid upper bound of the actual aliasing error.

To obtain an estimate of the truncation error, we require that  $g(t)$  approach zero at least as rapidly as  $\nu/|t|^\beta$  for  $t$  outside the sampled interval  $T$ , where  $\beta > 1.0$  and  $\nu$  is some

small constant. Mosely calculates the truncation error, denoted here by  $\gamma_2$ , to be given by

$$|\gamma_2| < \frac{|\nu|2^\beta}{(\beta-1)(T-\Delta)^{\beta-1}} \quad (\text{A6})$$

where  $\Delta = 1/SF$  is the interval between samples.

The total aliasing and truncation error in the DFT of the bandpass function  $g(t)$  is then,

$$|G_R(f) - G(f)| < \frac{|\epsilon|2^{\alpha+2}\zeta(\alpha)}{(SF)^\alpha} + \frac{|\nu|2^\beta}{(\beta-1)(T-\Delta)^{\beta-1}}. \quad (\text{A7})$$

The aliasing error may be reduced by choosing a larger sampling frequency  $SF$ , although in the bandpass case it must be an integral multiple of  $2W$ . The aliasing error could be further reduced if the data were filtered with a filter of steeper slope such that the signal in the bandwidth of interest was not affected. In this case the quantity  $\alpha$  would be larger. The truncation error may be reduced by choosing a longer analysis time  $T$ .

## APPENDIX B

## BEAMFORMED SPECTRUM ERROR

An estimate of the aliasing and truncation error present in the DFT of a bandpass function  $g(t)$  was determined in App. A. Here we examine the error in the beamformed spectrum  $G_R^{\text{beam}}(f)$ .

Consider a set of band-limited functions  $g^r(t)$  for  $r = 1, 2, \dots, p$ , all of which are identical after time shifting by  $t_r$ , where we let  $t_1 = 0$  be the reference channel chosen so that the remaining  $t_r$  are positive. Then,

$$\sum_{r=1}^p g^r(t-t_r) = pg^1(t). \quad (\text{B1})$$

If the set  $G^r(f)$  are the Fourier transforms of the  $g^r(t)$ , we have

$$\sum_{r=1}^p [e^{-2\pi i f t_r} G^r(f)] = pG^1(f). \quad (\text{B2})$$

We now sample each  $g^r(t)$  in the interval  $[-T/2, T/2]$  at a rate  $2W = N/T = 1/\Delta$  and beamform via the DFT using the method described in the main body of this report. Let  $G_R^r(f)$  be the set of spectra computed by the DFT's of the data before the beamforming is carried out. Then,

$$G_R^{\text{beam}}(f) = \sum_{r=1}^p [e^{-2\pi i f t_r} G_R^r(f)] \quad \text{and} \quad G^{\text{beam}}(f) = \sum_{r=1}^p [e^{-2\pi i f t_r} G^r(f)]$$

are the beam spectrum computed via the DFT and the ideal beam spectrum, respectively. Using Eq. (B2) and defining

$$\sum_{r=1}^p [e^{-2\pi i f t_r} G_R^r(f)] \equiv pG_R^1(f), \quad (\text{B3})$$

the difference between  $G_R^{\text{beam}}(f)$  and  $G^{\text{beam}}(f)$  is  $pG_R^1(f) - pG^1(f)$ . The factors of  $p$  are indicative of the signal gain that we expect from the beamforming process.

Moseley derives an equation for the truncation and aliasing error present in the DFT of a sampled low-pass time function  $g_n(t)$ . His error equation is expressed as a difference

between the spectrum  $G_R(f)$  computed via the DFT and the ideal continuous spectrum  $G(f)$ , where

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-2\pi i f t} dt$$

and

$$G_R(f) = \sum_{n=-N/2}^{(N/2)-1} [g(\Delta n) e^{-2\pi i f \Delta n}]$$

with  $\Delta = 1/SF = T/N$ , and  $SF$  is the sampling frequency.

A brief explanation of his method follows. By sampling  $g(t)$  with an infinite Dirac comb function, and taking the continuous transform, he obtains for  $|f| < 1/(2\Delta)$ , the true transform  $G(f)$  plus an infinite number of aliased spectral terms. Then he takes into account time truncation by subtracting samples of  $g(\Delta n)$  for  $n = N/2$  to  $\infty$  and obtains a series of error terms caused by truncation. Thus he obtains a measure of the aliasing and truncation error between  $G_R(f)$  and  $G(f)$ .

When we apply our beamforming procedure to this difference, we obtain

$$\sum_{r=1}^p \left\{ e^{-2\pi i f t_r} [G_R^r(f) - G^r(f)] \right\} = G_R^{\text{beam}}(f) - G^{\text{beam}}(f) = p G_R^1(f) - p G^1(f).$$

Thus the exponential multiplication and summing of the Moseley difference errors will give us a measure of the error between the DFT and ideal beam spectra. Doing this, we obtain

$$\begin{aligned} G_R^{\text{beam}}(f) - G^{\text{beam}}(f) &= \sum_{r=1}^p \left\{ \sum_{\ell=1}^{\infty} e^{-2\pi i f t_r} \left[ G^r \left( f - \frac{\ell}{\Delta} \right) + G^r \left( f + \frac{\ell}{\Delta} \right) \right] \right\} \\ &\quad - \Delta \sum_{r=1}^p \left\{ \sum_{n=N/2}^{\infty} g^r(n\Delta) e^{-2\pi i f (n\Delta + t_r)} \right. \\ &\quad \left. + \sum_{n=-\infty}^{-(N/2)-1} g^r(n\Delta) e^{-2\pi i f (n\Delta + t_r)} \right\}. \end{aligned} \quad (\text{B4})$$

The right-hand side of Eq. (B4) consists of two major terms which we'll denote as  $\gamma_1$  and  $\gamma_2$  and separately consider by virtue of

$$|G_R^{\text{beam}}(f) - G^{\text{beam}}(f)| = |\gamma_1 + \gamma_2| \leq |\gamma_1| + |\gamma_2|. \quad (\text{B5})$$

The first term, further reduced in Eq. (B6) below, represents the contribution to the beam-forming error caused by spectrum aliasing:

$$|\gamma_1| \leq \sum_{r=1}^p \left[ \sum_{\ell=1}^{\infty} \left( \left| G^r \left( f - \frac{\ell}{\Delta} \right) \right| + \left| G^r \left( f + \frac{\ell}{\Delta} \right) \right| \right) \right]. \quad (\text{B6})$$

Note that, because the multiplicative phase terms are simply phase shifts not affecting the amplitudes of the spectra, our upper bound estimate of the aliasing error is not affected at all by the introduction of the delays  $t_r$ .

As in App. A, assume that the  $G^r(f)$  each approach zero at least as rapidly as

$$\frac{\epsilon}{|f| - W_0 - (W/2)} \quad \text{for } |f| \text{ outside } [W_0, W_0 + W].$$

Then, except for a factor of  $p$ , the quantity  $\gamma_1$  is simply the aliasing error previously defined in App. A and is

$$|\gamma_1| < \frac{p|\epsilon|2^{\alpha+2}\zeta(\alpha)}{(SF)^{\alpha}}. \quad (\text{E7})$$

Recall that  $SF$  is equal to  $2W$  for the bandpass case, and  $\zeta(\alpha)$  is the Riemann zeta function.

The second term  $\gamma_2$  includes the effects of time domain truncation and cycling error. Let  $n^1 = n + t_r/\Delta$ . Then we have

$$|\gamma_2| = \left| \Delta \sum_{r=1}^p \left( \sum_{n^1=(N/2)+(t_r/\Delta)}^{\infty} g^r(n^1\Delta - t_r)e^{-2\pi i f n^1\Delta} + \sum_{n^1=-\infty}^{-(N/2)-1+t_r/\Delta} g^r(n^1\Delta - t_r)e^{-2\pi i f n^1\Delta} \right) \right|. \quad (\text{B8})$$



The multiplicative phase factors in the  $\gamma_2$  term manifest themselves as delays in time of the  $g^r(t)$ . Note that the limits on the inner sums have changed and we are now considering contributions to the error from outside the interval  $[-T/2+t_r, T/2+t_r]$ . Since the  $t_r$  were constrained to be positive, we may separate the sums over  $n^1$  into the following series (an analogous procedure applies for negative  $t_r$ 's):

$$|\gamma_2| = \left| \Delta \sum_{r=1}^p \left( \sum_{n^1=N/2}^{\infty} g^r(n^1 \Delta - t_r) e^{-2\pi i f n^1 \Delta} + \sum_{n^1=-\infty}^{-(N/2)-1} g^r(n^1 \Delta - t_r) e^{-2\pi i f n^1 \Delta} \right. \right. \\ \left. \left. - \sum_{n^1=N/2}^{(N/2)-1+t_r/\Delta} g^r(n^1 \Delta - t_r) e^{-2\pi i f n^1 \Delta} + \sum_{n^1=N/2}^{-(N/2)-1+t_r/\Delta} g^r(n^1 \Delta - t_r) e^{-2\pi i f n^1 \Delta} \right) \right|. \quad (\text{B9})$$

Interchanging sums over  $n^1$  and  $r$  and using Eq. (B1), we arrive at

$$|\gamma_2| \leq \Delta p \left| \sum_{n^1=N/2}^{\infty} g^1(n^1 \Delta) e^{-2\pi i f n^1 \Delta} + \sum_{n^1=-\infty}^{-(N/2)-1} g^1(n^1 \Delta) e^{-2\pi i f n^1 \Delta} \right| \\ + \Delta \left| \sum_{r=1}^p \left( \sum_{n^1=N/2}^{+(N/2)-1+t_r/\Delta} g^r(n^1 \Delta - t_r) e^{-2\pi i f n^1 \Delta} \right. \right. \\ \left. \left. - \sum_{n^1=-N/2}^{-(N/2)-1+t_r/\Delta} g^r(n^1 \Delta - t_r) e^{-2\pi i f n^1 \Delta} \right) \right|. \quad (\text{B10})$$

The first term on the right-hand side of Eq. (B10) is  $p$  times Moseley's truncation error. It is less than the quantity

$$\frac{p|\nu|2^\beta}{(\beta-1)(T-\Delta)^{\beta-1}} \quad (\text{B11})$$

for  $g(t) < \nu/|t|^\beta$ , where  $t$  is outside the interval  $[-T/2, T/2]$ ,  $\beta > 1.0$ , and  $\nu$  is some small constant (see App. A).

The second term, hereafter denoted by  $\gamma_2^1$ , is caused by the cyclic representation of  $g_n(t)$ . When a delay of  $g(n\Delta)$  is effected, a "cycling" takes place, resulting in a time interval where the delayed function is inexact. For example, for positive  $t_r$ , information in the interval  $[T/2 - t_r, T/2]$  is "lifted" and "inserted" into the interval  $[-T/2, -T/2 + t_r]$ . For small values of delay  $t_r$ , this error interval is small, as is the error itself. This can be seen in Eq. (B8) by noting that if  $t_r = 0$ , no additional terms are present to contribute to  $\gamma_2^1$ ; if  $t_r = \Delta$ , only one additional term contributes from each sum, etc.

In order to reduce and evaluate  $\gamma_2^1$ , note that

$$|\gamma_2^1| < \Delta \sum_{r=1}^p \left( \sum_{n^1=N/2}^{(N/2)-1+t_r/\Delta} |g^r(n^1\Delta - t_r)| + \sum_{n^1=-N/2}^{-(N/2)-1+t_r/\Delta} |g^r(n^1\Delta - t_r)| \right) \quad (\text{B12})$$

Now let  $g^r(t - t_r)$  approach zero at least as rapidly as

$$\frac{\nu}{|t|^\beta} \quad \text{for } T/2 < |t| < \infty$$

where  $\beta > 1.0$ , and  $\nu$  is some small constant. Recalling that  $g^r(t)$  approaches zero for  $t < -T/2$ , this is equivalent to letting  $g^r(t)$  approach zero for  $t$  outside the interval  $[-T/2, T/2 - t_r]$  for positive  $t_r$ . In order to maximize the error estimate, choose  $t_{\max}$  equal to the largest delay from the set  $t_r$ . Then,

$$|\gamma_2^1| < \frac{|\nu|p}{\Delta^{\beta-1}} \left( \sum_{n^1=N/2}^{(N/2)-1+t_{\max}/\Delta} (|n^1|)^{-\beta} + \sum_{n^1=(N/2)+1-t_{\max}/\Delta}^{N/2} (|n^1|)^{-\beta} \right) \quad (\text{B13})$$

Finally, since

$$\left. \begin{aligned} & \sum_{n^1=N/2}^{(N/2)-1+t_{\max}/\Delta} (|n^1|)^{-\beta} < \frac{t_{\max}}{\Delta} \left(\frac{N}{2}\right)^{-\beta} \\ & \text{and} \\ & \sum_{n^1=(N/2)+1-t_{\max}/\Delta}^{N/2} (|n^1|)^{-\beta} < \frac{t_{\max}}{\Delta} \left(\frac{N}{2} + 1 - \frac{t_{\max}}{\Delta}\right)^{-\beta} \end{aligned} \right\}, \quad (\text{B14})$$

we arrive at the expression

$$|\gamma_2^1| < \frac{|\nu| p t_{\max}}{\left(\frac{T}{2}\right)^\beta} \left(1 + \frac{1}{\left(1 + \frac{2}{N} - \frac{t_{\max}}{(T/2)}\right)^\beta}\right) \quad (\text{B15})$$

where we have used the relation  $N\Delta = T$ . It can be seen from Eq. (B15) that the cycling error may be decreased by choosing the analysis time  $T$  to be large.

The estimate of cyclic delaying error developed here is good for all practical situations where the maximum delay  $t_{\max}$  is much less than the total length  $T$  of signal being analyzed.\* In fact the model is good for delays up to half the analysis interval (see App. C), an extreme delay situation rarely, if ever, encountered with real arrays and signals. At  $t_{\max} = (T/2) + \Delta$ , the error term  $|\gamma_2^1|$  becomes infinite. This is because, for this particular delay, we require that  $g(t)$  be negligible outside the interval  $[-T/2, 0]$ . The function  $\nu/|t|^\beta$  has a singularity at  $t = 0$ , however, thus preventing negligibility of  $g(t)$ .

We may now add together the three error terms (aliasing, truncation, and cycling) to obtain

$$\begin{aligned} |G_R^{\text{beam}}(f) - G^{\text{beam}}(f)| &< \frac{p|\varepsilon|2^{\alpha+2}\zeta(\alpha)}{(SF)^\alpha} \\ &+ \frac{p|\nu|2^\beta}{(\beta-1)(T-\Delta)^{\beta-1}} + \frac{p|\nu|t_{\max}}{(T/2)^\beta} \left(1 + \frac{1}{\left(1 + \frac{2}{N} - \frac{t_{\max}}{(T/2)}\right)^\beta}\right). \end{aligned} \quad (\text{B16})$$

\*Recall from the section on Digital Bandpass Beamforming that a ratio of  $t_{\max}/T$  on the order of  $10^{-2}$  was considered typical.

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Thus, the error in the beamformed spectrum is the sum of three error terms: the first two are encountered in the bandpass spectrum analysis, with or without delay present, and are the aliasing and the truncation errors; the third term includes the error from the cyclic delays inherent in the beamforming process.

## APPENDIX C

## ILLUSTRATION OF THE METHOD

A computer program was written to illustrate the theory. A set of random numbers was generated using a CDC Library Function Subroutine.\* This set was then digital bandpass filtered for  $C = W_0/W = 12$  and for a sampling rate  $SF = 4(W_0 + W)$ , i.e., twice the usual Nyquist rate. The resulting set of numbers is taken to represent a redundant sampling of a continuous function  $g(t)$  bandlimited to the interval  $[W_0, W_0 + W]$ . The set can then be separated into two related interleaving sets  $\hat{g}_{n1}^{(1)}(t)$  and  $\hat{g}_{n1}^{(2)}(t)$ , either set completely determining  $g(t)$  (see Fig. C1). By choosing the starting point for  $\hat{g}_{n1}^{(1)}(t)$ , the set  $\hat{g}_{n1}^{(2)}(t)$  can be "delayed" from  $\hat{g}_{n1}^{(1)}(t)$  by odd multiples of  $1/SF$ . For the case shown here we let the delay  $t_0$  between the two sets be equal to  $13/SF = \Delta/2 = (1/2W)/2$ .

Both sets were "bandpass sampled" at a rate  $2W$  by retaining only every 13th value from each set, thus creating two sets of bandpass sampled data delayed from each other by  $13/SF$  (see Fig. C2);  $N$  was equal to 128, and Hanning weighting was used to minimize  $g(t)$  near  $t=0$  and  $t=T$ . Figure C3 shows a plot of the difference between the original Hanned set values before delaying. Note the large difference values, indicating that  $g_{n1}^{(1)}(t)$  and  $g_{n1}^{(2)}(t)$  were not at all coincident before being delayed.

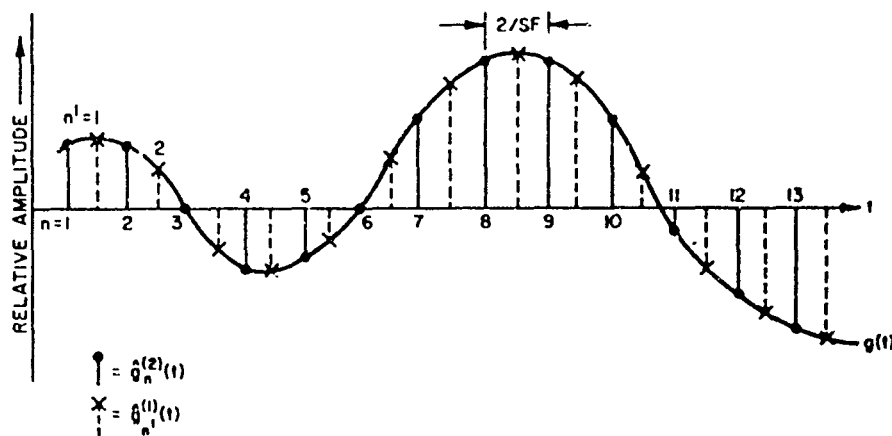


Fig. C1—A set of randomly generated numbers is digitally bandpass filtered and separated into two related interleaving sets  $\hat{g}_{n1}^{(1)}(t)$  and  $\hat{g}_{n1}^{(2)}(t)$  as shown here. Either set completely determines the bandpass function  $g(t)$ .

\*Spectrum analysis of the sequence before filtering resulted in a flat power spectrum, indicating that the series was random enough for our purposes.

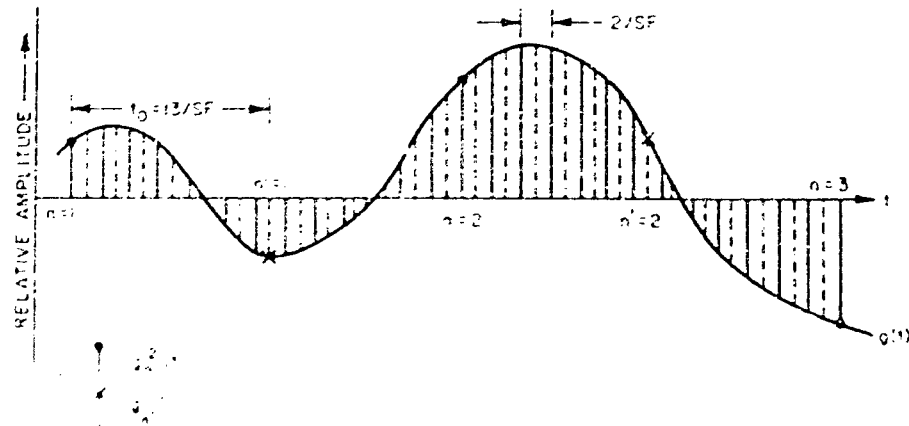


Fig. C2 The set  $\hat{g}_n^{(2)}(t)$  shown in Fig. C1 is delayed from  $\hat{g}_n^{(1)}(t)$  by an amount  $t_0 = 13/SF$ . Both sets are then "bandpass sampled" at a rate  $2B$  by retaining only every 13th value from each set. The resultant sets are then as shown here.

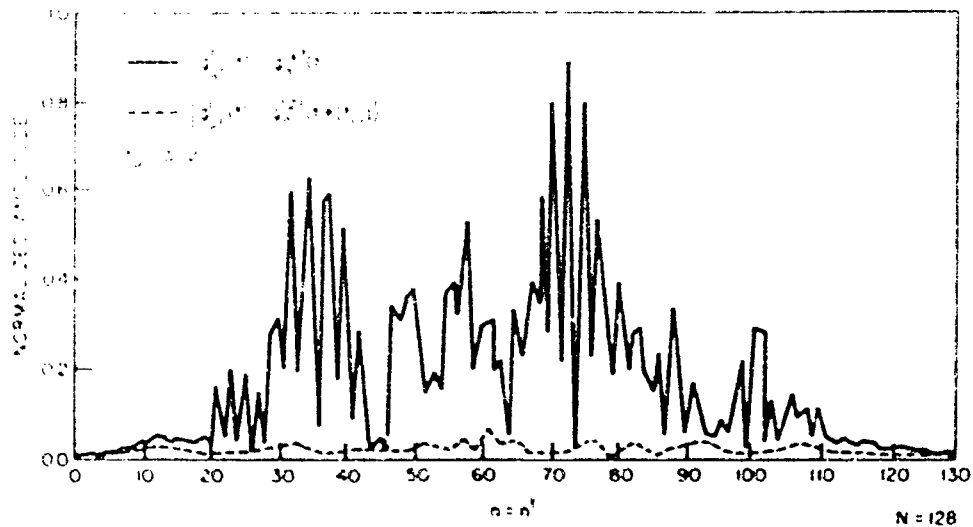


Fig. C3 -Amplitude differences between the original Hanned set values  $g_n^{(1)}(t)$  and  $g_n^{(2)}(t)$  before (top curve) and after (bottom curve) computing the advance function  $g_n^{(2)}(t+t_0)$ . The differences are large before delaying (top curve). After delaying, the two set values agree closely (bottom curve).

Equation (16) was programmed, for  $t_0 = -13/SF$ , and run on the  $g_n^{(2)}(t)$  data series. An IDFT of the resulting  $G_R^d(f_k')$  was computed, and the values of the "advanced" function  $g_R^{(2)}(t+|t_0|)$  were compared to those of  $g_{n1}^{(1)}(t)$ . A plot of the difference function  $g_{n1}^{(1)}(t) - g_R^{(2)}(t+|t_0|)$  is shown in Fig. C3. It is seen that the two series agree closely, having been properly delayed. The mean-square value of the "after-delay" difference function is equal to

$$\epsilon^1 = \frac{1}{N} \sum_{n=n^1=1}^N (g_{n1}^{(1)}(t) - g_R^{(2)}(t+|t_0|))^2 = 3 \times 10^{-4}.$$

Using Parseval's theorem which equates the mean-square value of a function to the sum over all frequencies of the square of the absolute value of its Fourier spectrum, and with the knowledge that the spectrum of the noise is approximately flat across the band, we compute the mean-square error per spectral estimate to be

$$2\epsilon^1/N = 5 \times 10^{-6} = -45 \text{ dB}$$

where the decibel numbers are referenced to the mean-square value of the original time series  $g_{n1}^{(1)}(t)$ . Using Eq. B16 we find that for our choice of filter, Hanning smoothing window, and time delay, the errors from aliasing, truncation, and cycling are predicted to be less than, respectively, -38 dB, -65 dB, and -106 dB. This indicates, then, that in this case almost all of the error is contributed by spectrum aliasing. There is almost no contribution to the error from the delay  $t_0 = \Delta/2$  (very small) or the truncation (the Hanning window is very small at the end points of the interval) problem.

The same two original data sets were again "bandpass sampled" and properly delayed, but this time the delay was very large. We let  $t_0 = (T/2) - (5\Delta/2)$ . This value of  $t_0$  is close to half the analysis interval  $T/2$ , at which point the limiting assumptions for negligibility of  $g(t)$  begin to break down. The computed variance per spectral estimate for this case was on the order of

$$2\epsilon^1/N = 10^{-3} = -22 \text{ dB}.$$

Again using Eq. B16, we predict the errors from aliasing, truncation, and cycling to be, respectively, -38 dB, -65 dB, and -20 dB. Clearly, the dominant error is from the cyclic delay term. This computed error agrees closely with the calculated experimental error and shows the adequacy of the analysis of App. B to handle this extreme delay case.